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On the holomorphic point of view in the theory of quantum knot invariants

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Abstract

In this paper we describe progress made toward the construction of the Witten–Reshetikhin–Turaev theory of knot invariants from a geometric point of view. This is done in the perspective of a joint result of the author with A. Uribe which relates the quantum group and the Weyl quantizations of the moduli space of flat $SU(2)$-connections on the torus. Two results are emphasized: the reconstruction from Weyl quantization of the restriction to the torus of the modular functor, and a description of a basis of the space of quantum observables on the torus in terms of colored curves, which answers a question related to quantum computing.

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1. Introduction

It is known that, for a compact simple Lie group G, the Hilbert space of the quantization of the moduli space of flat G-connections on a surface is the space of holomorphic sections of the Chern–Simons line bundle. Alternately, this space has a basis consisting of admissible colorings of the core of the handlebody bounded by the surface by irreducible representations of the quantum group of G.

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The quantization of the moduli space of flat $SU(2)$-connections on the *torus* was studied in detail in [7]. In that work two quantization models were compared, the equivariant Weyl quantization of the complex plane that covers the moduli space and the quantum group quantization, performed after Witten's ideas [12] with the techniques of Reshetikhin and Turaev [10,11]. There, it was shown that these two quantizations are *unitarily equivalent*.

This result gives rise to new possibilities for developing $SU(2)$ Chern–Simons theory, and in particular the study of the Jones polynomial [9], from a geometric point of view. The present paper describes some progress made in this direction. For other contributions to the subject see [1–3,5].

Here is a description of the contents of the paper. Section 2 gives a brief overview of the two quantization models. In Section 3 we show how the projective representation of the mapping class group of the torus, which arises in the Reshetikhin–Turaev topological quantum field theory, can be recovered from the Weyl quantization model. It is important to note that the Weyl quantization contains all the necessary information about the modular functor restricted to the torus. The next section contains explicit descriptions of quantum knot invariants as holomorphic sections of the Chern–Simons line bundle. Section 5 is devoted to the quantum observables. They are described as integral operators, much in the spirit of Witten's path integral, then their spectra are computed. The paper ends with an application to quantum computing. It determines the basis of the Hilbert space of ground states of a certain quantum system, which is the same as the vector space of the Chern–Simons theory of the quantum double of the group $SU(2)$. This solves a problem that arose in [6]. The author would like to thank the referee, whose comments considerably improved the quality of the manuscript.

2. The quantization of the moduli space of flat $SU(2)$-connections on the torus

The Hilbert space of the quantization of the moduli space of flat $SU(2)$-connections on the torus was described in [7]. Since in that paper a small error occurred in the exposition, but fortunately not in the final result, we briefly explain the construction again.

Denote by $\mathcal{M}$ the moduli space of flat $SU(2)$-connections on the torus $\mathbb{T}^2$, which is known informally as the "pillow case". $\mathcal{M}$ is the quotient of the complex plane by the symmetries $z \to z + m + ni$, $m, n \in \mathbb{Z}$, and $\sigma(z) = -z$. The symplectic form which determines the Poisson bracket and consequently the "classical mechanics" on $\mathcal{M}$ is $\omega = -\pi \mathrm{d}z \wedge \mathrm{d}\bar{z} = 2\pi i$ $\mathrm{d}x \wedge \mathrm{d}y$.

The Hilbert space of the quantization consists of holomorphic 1-forms on the smooth part of $\mathcal{M}$ with values in a line bundle of curvature ω^N, where $N = 1/\hbar$ is the reciprocal of Planck's constant. Constrains given by the Weil integrality condition and the Reshetikhin–Turaev theory require N to be an even integer, $N = 2r$. In Witten's theory, the number $r - 2$ is called the level of the quantization. The inner product is defined by integrating the cup product of two 1-forms over $\mathcal{M}$.

A 1-form on $\mathcal{M}$ can be written locally as $f(z)\mathrm{d}z$, where $f(z)$ is a section of the line bundle. This could almost be done globally, by lifting it to $\mathbb{C}$, except that on $\mathbb{C}$ the symmetry with respect to the origin σ changes $\mathrm{d}z$ to $-\mathrm{d}z$. This sign change can be incorporated into the line bundle. Consequently, the line bundle is determined by a cocycle χ on $\mathbb{C}$ satisfying

$$\chi(z, m + in) = \exp 2r\pi(-2inz + n^2), \qquad m, n \in \mathbb{Z},$$
$$\chi(z, \sigma) = -1.$$

The Hilbert space of the quantization is then identified with the space of odd theta functions

$$\mathcal{H}_r = \{f \in \mathcal{H}ol(\mathbb{C}) \mid f(z+m+in) = \mathrm{e}^{2r\pi(n^2-2inz)} f(z),\ f(z) = -f(-z)\}.$$

If we denote

$$\theta_j(z) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi(2rn^2+2jn)+2\pi iz(j+2rn)},$$

then the functions

$$\zeta_j(z) = \sqrt[4]{r}\mathrm{e}^{-\pi j^2/2r}(\theta_j(z) - \theta_{-j}(z)), \qquad j = 1, 2, \ldots, r-1$$

form an orthonormal basis of $\mathcal{H}_r$. For later use we extend the definition of $\zeta_j(z)$ to all integers j by the relations $\zeta_{r+j}(z) = -\zeta_{r-j}(z)$ and $\zeta_{-j}(z) = -\zeta_j(z)$. Note that $\zeta_0(z) = \zeta_r(z) = 0$.

The quantum observables are defined using Weyl quantization. For a classical observable $f \in C^\infty(\mathcal{M})$, the associated quantum observable is the Toeplitz operator with symbol $\mathrm{e}^{-\frac{\Delta}{8r}} f$. Here Δ is the Laplace operator $\frac{1}{2\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. We view Weyl quantization as interpolating between Wick and anti-Wick, as explained in [4].

The alternative quantization model, which appears in the realm of quantum groups [10], is the following. The orthonormal basis of the Hilbert space is formally identified with $V^j(\alpha)$, $j = 1, 2, \ldots, r-1$, the colorings of the core α of the solid torus by the j-dimensional irreducible representations of the quantum group of $SL(2,\mathbb{C})$ at a root of unity $q = \exp(2\pi i/r)$. The algebra of classical observables contains as a dense subset the ring generated by the traces of holonomies of $SU(2)$-connections along simple closed curves on the torus. The quantum observable associated to the trace in the k-dimensional irreducible representation of $SU(2)$ of such a holonomy along a curve γ is simply the curve γ colored by the k-dimensional irreducible representation of the quantum group of $SL(2,\mathbb{C})$ (with the usual convention when k is larger than r, the level of the quantization). There is a way of identifying colored curves with linear operators using knot invariants. For details we refer the reader to [7].

The main result of [7] is that the unitary map $\zeta_j(z) \to V^j(\alpha)$, $j = 1, 2, \ldots, r-1$, is an equivalence between the Weyl quantization and the quantum group quantization. In the process of proving this result we discovered a formula which will be used in what follows. Denote by $C(p,q)$, $p, q \in \mathbb{Z}$ the operator obtained by performing Weyl quantization with symbol $2\cos 2\pi(px+qy)$. Then

$$C(p,q)\zeta_m(z) = t^{-pq}(t^{2qm}\zeta_{m-p}(z) + t^{-2qm}\zeta_{m+p}(z)), \qquad m = 1, 2, \ldots, r-1,$$

where $t = \exp(\pi i/2r)$ (and so $t^4 = q$).

This paper describes some features of the Witten–Reshetikhin–Turaev theory from the analytical-geometric point of view. We will need an alternative formula for $\zeta_j(z)$. To obtain it we write $\zeta_j(z)$ as

$$\sqrt[4]{r}\mathrm{e}^{-\frac{\pi j^2}{2r}} \sum_{n=-\infty}^{\infty} (\mathrm{e}^{-\pi(2rn^2+2jn)+2\pi iz(j+2rn)} - \mathrm{e}^{-\pi(2rn^2-2jn)+2\pi iz(-j+2rn)}).$$

Note that

$$\begin{aligned}\mathrm{e}^{-\frac{\pi j^2}{2r}}\mathrm{e}^{-\pi(2rn^2+2jn)+2\pi iz(j+2rn)} &= \mathrm{e}^{-\frac{\pi j^2}{2r}}\mathrm{e}^{2\pi izj}\mathrm{e}^{-\pi(2rn^2+2jn)+4\pi irnz} \\ &= \mathrm{e}^{-2\pi rz^2}\mathrm{e}^{-2\pi r\left(n+\frac{j}{2r}-iz\right)^2}.\end{aligned}$$

Using the Poisson formula for e^{-x^2}, the first sum can be transformed into

$$e^{-2\pi r z^2}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{2r}} e^{2\pi i\left(\frac{j}{2r}-iz\right)n} = \sum_{n=-\infty}^{\infty} e^{-2\pi r z^2-\frac{\pi n^2}{2r}+2\pi n z} e^{\frac{\pi i j n}{r}}.$$

Replacing j by $-j$ we obtain that the other sum is equal to

$$\sum_{n=-\infty}^{\infty} e^{-2\pi r z^2-\frac{\pi n^2}{2r}+2\pi n z} e^{-\frac{\pi i j n}{r}}.$$

Subtracting the two and recalling the definition of the quantized integer

$$[n] = \frac{e^{\frac{\pi i n}{r}} - e^{-\frac{\pi i n}{r}}}{e^{\frac{\pi i}{r}} - e^{-\frac{\pi i}{r}}} = \frac{\sin\frac{n\pi}{r}}{\sin\frac{\pi}{r}} = \frac{t^{2n}-t^{-2n}}{t^2-t^{-2}}.$$

we obtain the following:

Lemma 2.1. *For $j = 1, 2, \ldots, r-1$,*

$$\zeta_j(z) = 2i\sqrt[4]{r}\sin\frac{\pi}{r}\sum_{n=-\infty}^{\infty} e^{-2\pi r\left(z-\frac{n}{2r}\right)^2}[nj].$$

As a corollary we find that the reproducing kernel of the Hilbert space $\mathcal{H}_r$ is

$$\begin{aligned} K(z,w) &= 2r^{\frac{3}{2}}\sum_{\substack{n,m=-\infty\\ 2r|m-n}}^{\infty} e^{-2r\pi\left[\left(z-\frac{n}{2r}\right)^2+\left(\overline{w}-\frac{m}{2r}\right)^2\right]} \\ &\quad - 2r^{\frac{3}{2}}\sum_{\substack{n,m=-\infty\\ 2r|m+n}}^{\infty} e^{-2r\pi\left[\left(z-\frac{n}{2r}\right)^2+\left(\overline{w}-\frac{m}{2r}\right)^2\right]} \\ &= 4r^{\frac{3}{2}}\sum_{k,m=-\infty}^{\infty} e^{-2r\pi\left[(z+k)^2+\frac{m^2}{4r^2}+\left(\overline{w}-\frac{m}{2r}\right)^2\right]}\sinh m\pi(z+k). \end{aligned}$$

This means that if $f(z)$ is an element of the Hilbert space of the quantization, namely an odd theta function, then $f(z) = \int_{\mathcal{M}} f(w)K(z,w)\mathrm{d}w$, for all $z \in \mathcal{M}$.

3. The projective representation of the mapping class group of the torus on the Hilbert space of the quantization

Part of the Reshetikhin–Turaev topological quantum field theory is a projective representation of the mapping class group of the torus onto the Hilbert space of the quantization. This representation is the restriction of the modular functor to the torus. We will show how this projective representation can be recovered from Weyl quantization.

A simple closed curve γ on the torus defines a smooth function on the moduli space of flat $SU(2)$-connections by taking the trace in the fundamental representation of $SU(2)$ of the holonomy along γ of the connection. Call this function f_γ and $\mathrm{op}(f_\gamma)$ the operator associated to it through Weyl quantization.

There is a right action of the mapping class group of the torus on $C^\infty(\mathcal{M})$. It is defined as follows. Identify the moduli space $\mathcal{M}$ with the algebraic variety of characters of $SU(2)$-representations of the fundamental group of the torus. If g is an element of the mapping class group of the torus and σ is an $SU(2)$-representation of the fundamental group of the torus, then

$$g \cdot \sigma : \quad \gamma \to \sigma(g(\gamma)),$$

We therefore have a left action of the mapping class group of the torus on the moduli space $\mathcal{M}$, which then induces a right action on $C^\infty(\mathcal{M})$ by $(f \cdot g)(x) = f(g \cdot x)$. Through Weyl quantization we obtain a right action on quantum observables.

In particular, if the element g of the mapping class group maps the curve γ to $g(\gamma)$, then $f_\gamma \cdot g = f_{g(\gamma)}$. In this sense we have a natural action of the mapping class group on symbols of operators and therefore a natural action on operators themselves.

If we view the torus as $\mathbb{R}^2/\mathbb{Z}^2$, then its mapping class group is generated by the maps $\mathcal{S}$ and $\mathcal{T}$ defined by $\mathcal{S}(x, y) = (-y, x)$, $\mathcal{T}(x, y) = (x, x + y)$. They act on quantum observables by

$$\mathrm{op}(f(x, y)) \cdot \mathcal{S} = \mathrm{op}(f(-y, x)), \ \mathrm{op}(f(x, y)) \cdot \mathcal{T} = \mathrm{op}(f(x, x + y)).$$

The Reshetikhin–Turaev topological quantum field theory comes with a projective representation of the mapping class group of the torus on $\mathcal{H}_r$ defined by $\rho(\mathcal{S}) = S$ and $\rho(\mathcal{T}) = T$, where

$$S = ([jk])_{1\leq j,k\leq r-1}\,, \qquad \text{and} \qquad T = (\delta_{j,k} t^{j^2-1})_{1\leq j,k\leq r-1}.$$

Here, as before, $[jk]$ is the quantized integer, $t = \mathrm{e}^{\frac{i\pi}{2r}}$, and $\delta_{j,k}$ is the Kronecker symbol (as we are only interested in projective representations we did not incorporate the factor $1/X$, with $X = \sqrt{\sum_{k=1}^{r-1}[k]^2}$, in the definition of S).

In order for the entire theory to be consistent, this representation must be compatible with the natural action on the algebra of quantum observables, which means that if $\mathrm{op}(f)$ is a quantum observable and g and element of the mapping class group, then

$$\mathrm{op}(f) \cdot g = \rho(g)^{-1} \mathrm{op}(f) \rho(g).$$

The next result shows that Weyl quantization together with this condition determine the projective representation of the mapping class group.

Theorem 3.1. *There is a unique projective representation of the mapping class group of the torus on the Hilbert space of the quantization which is compatible with the natural action of the mapping class group on quantum observables, and this is the projective representation from the Reshetikhin–Turaev theory.*

Proof. Let $S = (a_{k,j})_{1\leq k,j\leq r-1}$ and $T = (b_{k,j})_{1\leq k,j\leq r-1}$ be the S- and T-matrices of such a projective representation.

First, let us extend the definition of $a_{k,j}$ so that the indices can be any integer numbers. The equalities

$$\sum_{k=1}^{r-1} a_{k,2r-j}\zeta_k(z) = S\zeta_{2r-j}(z) = S(-\zeta_j(z)) = -S\zeta_j(z) = -\sum_{k=1}^{r-1} a_{k,j}\zeta_k(z)$$

$$\sum_{k=1}^{r-1} a_{k+r,j}\zeta_{k+r}(z) = S\zeta_j(z) = \sum_{k=1}^{r-1} a_{k,j}\zeta_k(z)$$

show that the correct choice for $a_{k,j}$ is as odd and periodic in j and k of period $2r$, that is $a_{k,-j} = a_{-k,j} = -a_{k,j}$. Moreover, $\zeta_0 = 0$ means that we can choose $a_{k,0} = a_{0,j} = 0$. The same conventions apply for $b_{k,j}$.

Recall that $C(p,q)$, $p, q \in \mathbb{Z}$ denotes the operator with symbol $2\cos 2\pi(px + qy)$. We will use the previously mentioned formula

$$C(p,q)\zeta_m(z) = t^{-pq}(t^{2qm}\zeta_{m-p}(z) + t^{-2qm}\zeta_{m+p}(z)).$$

Let us look at the action of S on the quantum observables $C(p,q)$, $p, q \in \mathbb{Z}$. The equality $S^{-1}C(p,q)S = C(-q,p)$ implies

$$C(p,q)S\zeta_j(z) = SC(-q,p)\zeta_j(z), \qquad j = 1, 2, \ldots, r-1.$$

At this point we need to make sure that we are able to shift indices in the summation, and for that we have to let these indices range between 1 and $2r$ (and not just between 1 and $r-1$). To this end we write $S\zeta_j(z) = \sum_{k=1}^{2r} \frac{1}{2} a_{k,j}\zeta_k(z)$, $j = 1, 2, \ldots, 2r$. This is no longer an expansion in the basis of the Hilbert space, as each element of the basis appears twice. Consequently, for a fixed j we have

$$\sum_{k=1}^{2r} \left(t^{-pq+2q(k+p)} a_{k+p,j} + t^{-pq-2q(k-p)} a_{k-p,j}\right) \zeta_k(z)$$
$$= \sum_{k=1}^{2r} \left(t^{pq+2pj} a_{k,j+q} + t^{pq-2pj} a_{k,j-q}\right) \zeta_k(z).$$

Both sides of the equality are antisymmetric under $k \to 2r - k$. For this reason we can equate the coefficients of $\zeta_k(z)$ to obtain that for any p, q, k, j,

$$t^{2qk} a_{k+p,j} + t^{-2qk} a_{k-p,j} = t^{2pj} a_{k,j+q} + t^{-2pj} a_{k,j-q}.$$

Setting $p = 0$, $q = k = 1$ we obtain the recursive relation

$$a_{1,j+1} = (t^2 + t^{-2})a_{1,j} - a_{1,j-1}.$$

Since we are looking for a projective representation, we can set $a_{1,1} = 1$, which combined with $a_{1,0} = 0$ yields $a_{1,j} = [j]$.

Also, setting $q = 0$, $p = 1$, we obtain the recursive relation

$$a_{k+1,j} + a_{k-1,j} = (t^{2j} + t^{-2j})a_{k,j},$$

whence inductively we obtain $a_{k,j} = [kj]$, as desired.

Let us study the T-matrix now. Similarly

$$C(p,q)T\zeta_j(z) = TC(p,q+p)\zeta_j(z), \qquad j = 1, 2, \ldots, r-1.$$

Again we extend the indices to the full range 1 through $2r$ to be able to shift indices in the summation, and write the above equality in expanded form as

$$\sum_{k=1}^{2r} (t^{-pq+2q(k+p)} b_{k+p,j} + t^{-pq-2q(k-p)} b_{k-p,j})\zeta_k(z)$$
$$= \sum_{k=1}^{2r} (t^{-p(q+p)+2(q+p)j} b_{k,j-p} + t^{-p(q+p)-2(q+p)j} b_{k,j+p})\zeta_k(z).$$

Hence for any p, q, k, j,

$$t^{2qk+2pq}b_{k+p,j} + t^{-2qk+2pq}b_{k-p,j} = t^{-p^2+2qj+2pj}b_{k,j-p} + t^{-p^2-2qj-2pj}b_{k,j+p}.$$

For $p = 0$ we obtain

$$(t^{2qk} + t^{-2qk})b_{k,j} = (t^{2qj} + t^{-2qj})b_{k,j}.$$

This implies that $b_{k,j} = 0$ if $k \neq j$; therefore T is diagonal. Setting $p = 1, k = j - 1$ we obtain

$$b_{j,j} = t^{2j-1}b_{j-1,j-1}, \qquad j = 1, 2, \ldots, r-1.$$

Again, since we are looking for a projective representation, we are allowed to choose $b_{1,1} = 1$, in which case we obtain inductively $b_{j,j} = t^{j^2-1}$, $j = 1, 2, \ldots, r-1$, as desired. □

We stress again that this theorem shows how the well known projective representation of the mapping class group of the torus on the Hilbert space can be introduced naturally using Weyl quantization.

It is time now to describe the action of S and T on the vectors of the basis. There is nothing to discuss about T since it is diagonal. For S we have

Proposition 3.2. *The action of S on the basis $\zeta_m(z)$, $m = 1, 2, \ldots, r-1$ is given by*

$$S\zeta_m(z) = 2i\sqrt{2}r^{\frac{3}{4}}\mathrm{e}^{\frac{-\pi m^2}{2r}} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-2\pi r(z-k)^2} \sinh 2\pi m(z-k).$$

Proof. We have $S\zeta_m(z) = \frac{1}{X}\sum_{j=1}^{r-1}[jm]\zeta_j(z)$. By Lemma 2.1 this is equal to

$$\frac{2i\sqrt{2}}{\sqrt[4]{r}} \sin^2 \frac{\pi}{r}\mathrm{e}^{-2\pi r z^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi n z - \frac{\pi n^2}{2r}} \sum_{j=1}^{r-1}[nj][jm].$$

We compute $\sum_{j=1}^{r-1}[nj][jm]$, which is

$$(t^2 - t^{-2})^{-2}\sum_{j=0}^{r-1}\left(\mathrm{e}^{\frac{\pi i n j}{r}} - \mathrm{e}^{-\frac{\pi i n j}{r}}\right)\left(\mathrm{e}^{\frac{\pi i j m}{r}} - \mathrm{e}^{-\frac{\pi i j m}{r}}\right)$$
$$= (t^2 - t^{-2})^{-2} \sum_{-r+1\leq j\leq r-1}\left(\mathrm{e}^{\frac{\pi i j(n+m)}{r}} - \mathrm{e}^{\frac{\pi i j(n-m)}{r}}\right).$$

But

$$\sum_{-r+1\leq j\leq r-1}\left(\mathrm{e}^{\frac{\pi i k}{r}}\right)^j = \begin{cases} 2r-1 & \text{if } 2r \text{ divides } k \\ -(-1)^k & \text{otherwise.} \end{cases}$$

Therefore the sum we are computing is equal to

$$2r\sum_{n,2r|n+m} \mathrm{e}^{-2\pi r z^2 - \frac{\pi n^2}{2r} + 2\pi n z} - 2r\sum_{n,2r|n-m} \mathrm{e}^{-2\pi r z^2 - \frac{\pi n^2}{2r} + 2\pi n z}$$

multiplied by $ir^{-\frac{1}{4}}/\sqrt{2}$. Writing $n = 2rk \pm m$ we obtain the formula from the statement. □

4. Knot and link invariants as holomorphic sections

In the Reshetikhin–Turaev theory, the quantum invariant of a knot, viewed as a vector in the Hilbert space associated to the torus, is expressed as

$$Z(K) = \frac{1}{X}\sum_{k=1}^{r-1} J(K,j)V^j(\alpha),$$

where $X = \sqrt{\sum_{k=1}^{r-1}[k]^2}$, $J(K,j)$ is the jth colored Jones polynomial of K, and $V^j(\alpha)$ is the orthonormal basis consisting of colorings of the core of the solid torus by irreducible representations. As explained in Section 2, the result from [7] allows us to identify the elements of this orthonormal basis with holomorphic sections of the Chern–Simons line bundle. We therefore have

Proposition 4.1. *The quantum invariant in level r of a knot K is the holomorphic section of the Chern–Simons line bundle over $\mathcal{M}$ defined by the formula*

$$Z(K) = 2\sqrt{2}ir^{-\frac{1}{4}}\sin^2\frac{\pi}{r}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi r\left(z-\frac{n}{2r}\right)^2}\sum_{j=1}^{r-1}[nj]J(K,j),$$

where $J(K,j)$ is the jth colored Jones polynomial of K.

Proof. It follows from Lemma 2.1 since

$$Z(K) = \frac{1}{X}\sum_{j=1}^{r-1} J(K,j)\zeta_j(z). \quad \square$$

Example 1. The quantum invariant of the trivial knot is

$$Z(0) = 2i\sqrt{2}r^{\frac{3}{4}}\mathrm{e}^{-\frac{\pi}{2r}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-2\pi r(z-n)^2}\sinh 2\pi(z-n).$$

Because $J(0,j) = [j]$, the formula is a particular case of Proposition 3.2, since $Z(0) = S\zeta_1(z)$.

Example 2. The quantum invariant of the (p,q)-torus knot is

$$Z(K_{p,q}) = -\frac{1}{\sqrt{2}}r^{-\frac{1}{4}}\sin\frac{\pi}{r}\sum_{n=-\infty}^{\infty} C_n\mathrm{e}^{-2\pi r\left(z-\frac{n}{2r}\right)^2},$$

where

$$C_n = \frac{1}{\sin\frac{n\pi}{r}}\sum_{k=1}^{r-1} t^{-pqk^2}\left(\left[2n\left\lfloor\frac{r-1-k}{2}\right\rfloor + kn + n\right] - [kn-n]\right) \\ \times\left([kp+kq+1]-[kp-kq+1]\right).$$

In this formula square brackets represent quantized integers while $\lfloor\cdot\rfloor$ represents the greatest integer function.

This is a consequence of the formula for the jth colored Jones polynomial of a torus knot deduced in [8]:

$$J(K_{p,q}, j) = \sum_{\substack{0 \le k \le j \\ k \equiv j \, (\mathrm{mod}\ 2)}} \frac{t^{-pqk^2}}{t^2 - t^{-2}} ([kp + kq + 1] - [kp - kq + 1]).$$

Explicitly

$$\begin{aligned}
Z(K_{p,q}) &= 2\sqrt{2} i r^{-\frac{1}{4}} \sin^2 \frac{\pi}{r} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi r (z - \frac{n}{2r})^2} \sum_{j=1}^{r-1} [nj] J(T_{p,q}, j) \\
&= -\frac{i}{\sqrt{2}} r^{-\frac{1}{4}} \sin \frac{\pi}{r} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi r (z - \frac{n}{2r})^2} \\
&\quad \times \sum_{j=1}^{r-1} [nj] \sum_{\substack{0 \le k \le j \\ k \equiv j \, (\mathrm{mod}\ 2)}} t^{-pqk^2} ([kp + kq + 1] - [kp - kq + 1]).
\end{aligned}$$

Changing the order of summation in the double sum we find that it is equal to

$$\begin{aligned}
&\sum_{k=1}^{r-1} t^{-pqk^2} ([kp + kq + 1] - [kp - kq + 1]) \sum_{\substack{k \le j \le r-1 \\ j \equiv k \, (\mathrm{mod}\ 2)}} [nj] \\
&\quad = \sum_{k=1}^{r-1} t^{-pqk^2} ([kp + kq + 1] - [kp - kq + 1]) \sum_{0 \le m \le \lfloor \frac{r-1-k}{2} \rfloor} [n(2m + k)].
\end{aligned}$$

Write the quantized integer in explicit form, then sum the exponentials as geometric series to obtain that the inside sum is equal to

$$\begin{aligned}
&\mathrm{e}^{-\frac{n\pi i}{r}} \frac{\exp\left(\frac{n\pi i}{r}\left(2\left(\left\lfloor \frac{r-1-k}{2} \right\rfloor + 1\right) + k\right)\right) - \exp\left(\frac{nk\pi i}{r}\right)}{\exp\left(\frac{n\pi i}{r}\right) - \exp\left(-\frac{n\pi i}{r}\right)} \\
&\quad - \mathrm{e}^{\frac{n\pi i}{r}} \frac{\exp\left(-\frac{n\pi i}{r}\left(2\left(\left\lfloor \frac{r-1-k}{2} \right\rfloor + 1\right) + k\right)\right) - \exp\left(-\frac{nk\pi i}{r}\right)}{\exp\left(\frac{n\pi i}{r}\right) - \exp\left(-\frac{n\pi i}{r}\right)}
\end{aligned}$$

everything multiplied by a factor of $(\mathrm{e}^{\frac{\pi i}{r}} - \mathrm{e}^{-\frac{-\pi i}{r}})^{-1}$. Using the definition of quantized integers, we find that this is equal to C_n.

Because of the presence of the greatest integer function, the formula cannot be further simplified using a Gauss sum.

For a link L with k components, the Hilbert space is obtained by taking the tensor product of k copies of the Hilbert space of the torus. The formula for the quantum invariant is then

$$\frac{1}{X} \sum_{j_1, j_2, \ldots, j_k = 1}^{r-1} J(L, j_1, j_2, \ldots, j_k) V^{j_1}(\alpha) \otimes V^{j_2}(\alpha) \otimes \cdots \otimes V^{j_k}(\alpha).$$

This can again be translated to the analytical setting replacing $V^j(\alpha)$'s by $\zeta_j(z)$'s. Here is one example.

Example 3. The quantum invariant of the Hopf link is

$$Z(L) = -r\sqrt{2}\sin\frac{\pi}{r}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}[nk]\mathrm{e}^{-2r\pi\left[\left(z-\frac{n}{2r}\right)^2+\left(w-\frac{k}{2r}\right)^2\right]}.$$

Indeed,

$$Z(L) = \frac{1}{X}\sum_{j,m=1}^{r-1}[jm]\zeta_j(z)\zeta_m(w).$$

Using Lemma 2.1 we transform this into

$$-\frac{i}{2\sqrt{2}}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-2r\pi\left[\left(z-\frac{n}{2r}\right)^2+\left(w-\frac{k}{2r}\right)^2\right]}\sum_{m,j=1}^{r-1}\left(\mathrm{e}^{\frac{\pi inj}{r}}-\mathrm{e}^{-\frac{\pi inj}{r}}\right)$$
$$\times\left(\mathrm{e}^{\frac{\pi ijm}{r}}-\mathrm{e}^{-\frac{\pi ijm}{r}}\right)\left(\mathrm{e}^{\frac{\pi imk}{r}}-\mathrm{e}^{-\frac{\pi imk}{r}}\right).$$

A computation with roots of unity similar to the one from the proof of Proposition 3.2 shows that the innermost double sum is equal to

$$-2r\left(\mathrm{e}^{\frac{\pi ink}{r}}-\mathrm{e}^{-\frac{\pi ink}{r}}\right)$$

and the formula follows.

5. Some properties of the quantum observables

In the Feynman path integral formulation, the operator $C(p,q)$ representing the quantization of the function $2\cos 2\pi(px+qy)$ is the integral operator with kernel

$$\mathcal{K}_{p,q}(A_1,A_2) = \int_{\mathcal{M}_{A_1,A_2}}\mathrm{e}^{iN\mathcal{L}(A)}(\mathrm{tr}_{V^{n+1}}-\mathrm{tr}_{V^{n-1}})(\mathrm{hol}_C(A))\mathcal{D}A.$$

Here A_1, A_2 are conjugacy classes of connections on the torus $\mathbb{T}^2$ modulo gauge transformations, A is a conjugacy class of connections on $\mathbb{T}^2\times[0,1]$ modulo gauge transformations such that $A\big|_{\mathbb{T}^2\times\{0\}} = A_1$ and $A\big|_{\mathbb{T}^2\times\{1\}} = A_2$, n is the greatest common divisor of p and q, and $\mathrm{tr}_{V^n}(\mathrm{hol}_C(A))$, known as the Wilson line, is the trace of the n-dimensional irreducible representation of $SU(2)$ evaluated on the holonomy of A around the curve C of slope p/q. The "integral" is taken over all conjugacy classes of connections A.

We now exhibit a mathematically well defined formula for this kernel. In complex coordinates, the kernel is given by

$$\mathcal{K}_{p,q}(z,w) = \sum_{j=1}^{r-1}\left(C(p,q)\zeta_j\right)(z)\overline{\zeta_j(w)}.$$

A straightforward computation shows that

Proposition 5.1. *The kernel of the operator $C(p,q)$ is given by*

$$\mathcal{K}_{p,q}(z,w) = 2r^{\frac{3}{2}}\sum_{\substack{m,n=-\infty\\ 2r|q\pm(n-m)}}^{\infty}\mathrm{e}^{-2r\pi\left[\left(z-\frac{n}{2r}\right)^2+\left(\overline{w}-\frac{m}{2r}\right)^2\right]\mp\frac{npi\pi}{r}}$$

$$-2r^{\frac{3}{2}} \sum_{\substack{m,n=-\infty \\ 2r|q\pm(n+m)}}^{\infty} \mathrm{e}^{-2r\pi\left[(z-\frac{n}{2r})^2+(\overline{w}-\frac{m}{2r})^2\right]\mp\frac{npi\pi}{r}}.$$

The operator $C(p,q)$ acts on theta functions in the Hilbert space $\mathcal{H}_r$ by

$$(C(p,q)f)(z) = \int_{\mathcal{M}} \mathcal{K}_{p,q}(z,w)f(w)\mathrm{d}w.$$

Clearly $K_{p,q}(z,w)$ is holomorphic in z and antiholomorphic in w.

Proposition 5.2. *The characteristic polynomial of the operator $C(p,q)$ is*

$$\prod_{k=1}^{r-1}\left(\lambda - 2\cos\frac{\gcd(p,q,2r)k\pi}{r}\right).$$

Proof. Note that if $p = np'$, $q = nq'$, with p', q' coprime, then $C(p,q) = T_n(C(p',q'))$, where $T_n(x)$ is the nth Chebyshev polynomial (subject to the normalization $T_0(x) = 2$, $T_1(x) = x$, $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$, $n \geq 1$). So the case where p and q have a common divisor follows from the case where they are coprime via the spectral mapping theorem. Let us assume that p and q are coprime. As a consequence of Theorem 3.1, there exists an invertible matrix A such that $C(p,q) = A^{-1}C(1,0)A$. In fact $A = \rho(g)$, where g is the element of the mapping class group that maps the $(1,0)$ curve on the torus to the (p,q) curve. Because the characteristic polynomial is invariant under conjugation, it suffices to prove the property for $C(1,0)$. It is easy to check that the characteristic polynomials in dimensions $r+1$, r, and $r-1$ are related by $p_{r+1}(\lambda) = \lambda p_r(\lambda) - p_{r-1}(\lambda)$. Also $p_1(\lambda) = \lambda$ and $p_2(\lambda) = \lambda^2 - 1$. It follows that $p_r(\lambda) = S_r(\lambda)$, where $S_n(\lambda)$ denotes the Chebyshev polynomial of second type, $S_0(\lambda) = 1$, $S_1(\lambda) = \lambda$, $S_{n+1}(\lambda) = \lambda S_n(\lambda) - S_{n-1}(\lambda)$, $n \geq 1$. Factoring $S_r(\lambda)$ we obtain the desired formula. □

As a corollary, we recover in analytical setting the well known topological fact that a simple closed curve on the torus colored by the r-dimensional irreducible representation of the quantum group of $SL(2,\mathbb{C})$ is equal to zero, i.e., the operator with symbol equal to the trace of the holonomy along a curve in the r-dimensional irreducible representation of $SU(2)$ is the zero operator.

6. An application to quantum computing

In [6] the authors analyzed a possible quantum system suitable for quantum computation which is based on the fractional quantum Hall effect. The model we have in mind happens at the plateau corresponding to the fraction 12/5, where a non-abelian statistics has been predicted. The subspace of ground states of the Hilbert space of the quantum system can be identified with the vector space of an $SU(2)_r \times \overline{SU(2)}_r$ Chern–Simons quantum field theory for $r = 5$. This in turn can be obtained through Drinfeld's quantum double construction, or can be simply identified with the linear space of operators (quantum observables) of the $SU(2)_r$ Chern–Simons theory. The authors considered the case of the torus and were particularly interested in finding a basis of this vector space in terms of curves on the torus colored by representations of the quantum group of $SU(2)$. They succeeded for the case $r = 3$, but the real goal was $r = 5$. In this section we will answer their question for an arbitrary r. The problem was bought to our attention by Zh. Wang.

It is known that the vector space in discussion has dimension $(r-1)^2$ and is generated by the operators $C(p,q)$, $p,q \in \mathbb{Z}$ introduced before. Recall that if n denotes the greatest common divisor of p and q, and $p' = p/n$, $q' = q/n$, then $C(p,q)$ can be identified with the curve of slope p'/q' on the torus colored by the difference of the $n+1$st and $n-1$st dimensional irreducible representations of $SU(2)$. Our goal is to find a basis consisting of operators of the form $C(p,q)$. The key idea is to work in the basis $\zeta_j(z)$, $j = 1,2,\ldots,r-1$, and use the formula

$$C(p,q)\zeta_m(z) = t^{-pq}(t^{2qm}\zeta_{m-p}(z) + t^{-2qm}\zeta_{m+p}(z)).$$

We see now that in the basis $\zeta_j(z)$, $j = 1,2,\ldots,r-1$ the matrices of the operators $C(p,q)$ are particularly simple. To summarize our approach, through a linear isomorphism we identify the Hilbert space with a space of operators, then choose a basis in which the matrices of these operators are simple enough. We obtain

Theorem 6.1. *A basis of the linear space of quantum observables of the $SU(2)_r$ Chern–Simons theory on the torus is given by the operators*

$$\begin{aligned} &C(0,q), \quad 0 \le q \le r-2,\\ &C(p,q), \quad 1 \le p \le r-2, -r+p+2 \le q \le r-p-1. \end{aligned}$$

Proof. We will show that the $C(p,q)$, with p,q ranging as described in the statement, span the space of $(r-1)\times(r-1)$ matrices. Start with the diagonal.

Lemma 6.2. *The diagonal matrices are spanned by $C(0,q)$, $0 \le q \le r-2$.*

Proof. The matrix of $C(0,q)$ is the diagonal matrix with entries

$$\left(\cos\frac{q\pi}{r}, \cos\frac{2q\pi}{r}, \ldots, \cos\frac{(r-1)q\pi}{r}\right).$$

Denote $\alpha = \frac{\pi}{r}$. We have to show that the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \cos\alpha & \cos 2\alpha & \cdots & \cos(r-1)\alpha \\ \cos 2\alpha & \cos 4\alpha & \cdots & \cos 2(r-1)\alpha \\ \cdots & \cdots & \ddots & \cdots \\ \cos(r-2)\alpha & \cos 2(r-2)\alpha & \cdots & \cos(r-1)(r-2)\alpha \end{vmatrix}$$

is nonzero. To compute the determinant, let $x_1 = 2\cos\alpha$, $x_2 = 2\cos 2\alpha, \ldots, x_{r-1} = 2\cos(r-1)\alpha$. Denote by $T_n(x)$ the nth Chebyshev polynomial ($T_0(x) = 2, T_1(x) = x, T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$). Then the determinant is

$$\frac{1}{2^{r-1}}\begin{vmatrix} T_0(x_1) & T_0(x_2) & \cdots & T_0(x_{r-1}) \\ T_1(x_1) & T_1(x_2) & \cdots & T_1(x_{r-1}) \\ \cdots & \cdots & \ddots & \cdots \\ T_{r-2}(x_1) & T_{r-2}(x_2) & \cdots & T_{r-2}(x_{r-1}) \end{vmatrix}.$$

Row operations transform this into the Vandermonde determinant. We conclude that the value of the original determinant is

$$\frac{1}{2^{r-1}} \prod_{1\le k<j\le r-1} \left(\cos\frac{j\pi}{r} - \cos\frac{k\pi}{r}\right) \neq 0. \quad \square$$

Let us return to the proof of the theorem. For some nonzero $k \le r-2$, let us look at those $C(p,q)$ with $0 \le p \le k$. The nonzero entries of the matrix of such an element lie at distance at most k from the main diagonal, i.e., they are among the a_{ij}'s with $i-k \le j \le i+k$.

We prove by induction on p that $C(k,q)$ with q subject to the conditions from the statement and $0 \le k \le p$ span M_p, the set of all matrices whose only nonzero elements are of the form a_{ij}, with $i-p \le j \le i+p$. The base case $p=0$ was proved in the lemma.

Assume that the property is true for $p-1$, and let us prove it for p. Consider the matrix of $t^{pq}C(p,q)$. Using the inductive hypothesis we can add to it an element of M_{p-1}, so that the resulting matrix $A_{p,q}$ is of the form

$$\begin{pmatrix}
0 & 0 & \cdots & t^{-2q} & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & t^{-4q} & \cdots & 0\\
\cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & \cdots & 0 & 0 & \cdots & t^{-2(r-1-p)q}\\
\cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots\\
t^{2(p+1)q} & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & t^{2(p+2)q} & \cdots & 0 & 0 & \cdots & 0\\
\cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots
\end{pmatrix}.$$

The nonzero entries of $A_{p,q}$ are those of indices $(1,p+1)$, $(2,p+2)$, $\ldots$, $(r-p-1,r-1)$, $(p+1,1)$, $(p+2,2)$, $\ldots$, $(r-1,r-p-1)$ (those at distance p from the main diagonal of the matrix). The space $M_p \ominus M_{p-1}$ has dimension $2r-2k-2$, and a basis $E_{i,j}$, $(i,j) \in \{(1,p+1),(2,p+2),\ldots\}$, where $E_{i,j}$ denotes the matrix whose only nonzero entry is equal to 1 and is that of index (i,j). In this basis the coordinates of $A_{p,q}$ are

$$\left(t^{(-2)q}, t^{(-4)q}, \ldots, t^{-2(r-p-1)q}, t^{2(p+1)q}, \ldots, t^{2(r-1)q}\right).$$

To show that $A_{p,q}$, $-r+p+2 \le q \le r-p-1$, form a basis of $M_p \ominus M_{p-1}$ we arrange the entries of these vectors in a determinant, and show that this determinant is not equal to 0. With the convention $x_1 = t^{-2}$, $x_2 = t^{-4}$, $x_3 = t^{-6}$, $\ldots$, the determinant is

$$\begin{vmatrix}
x_1^{-r+p+2} & x_2^{-r+p+2} & \cdots & x_{2r-2p-2}^{-r+p+2}\\
x_1^{-r+p+3} & x_2^{-r+p+3} & \cdots & x_{2r-2p-2}^{-r+p+3}\\
\cdots & \cdots & \ddots & \cdots\\
x_1^{r-p-1} & x_2^{r-p-1} & \cdots & x_{2r-2p-2}^{r-p-1}
\end{vmatrix}.$$

Multiplying this determinant by

$$(x_1 x_2 \cdots x_{2r-2p})^{r-p-2}$$

produces a Vandermonde determinant, which is nonzero since the x_i's are distinct. This completes the inductive argument, and consequently the proof of the theorem. $\square$

We point out that contrary to a naive intuition, the indices of the basis elements do not range in an $(r-1) \times (r-1)$ rectangle, but in a triangular region, a surprising fact already observed in [6] for $r = 3$. For $r = 5$, we would like to describe a basis more in the spirit of the above-mentioned paper. For that let us denote by $V^k(m, n)$ the curve of slope m/n on the torus colored by the k-dimensional irreducible representation of the quantum group of $SL(2, \mathbb{C})$.

Corollary 1. *The linear space of quantum observables for $r = 5$ has a basis formed by the identity operator together with the operators* $V^2(0, 1)$, $V^3(0, 1)$, $V^4(0, 1)$, $V^2(1, -2)$, $V^2(1, -1)$, $V^2(1, 0)$, $V^2(1, 1)$, $V^2(1, 2)$, $V^2(1, 3)$, $V^2(2, -1)$, $V^3(1, 0)$, $V^2(2, 1)$, $V^3(1, 1)$, $V^4(1, 0)$, $V^2(3, 1)$.

Proof. This follows from the theorem using the identity

$$C(p, q) = V^{n+1}(p', q') - V^{n-1}(p', q'),$$

where n is the greatest common divisor of p and q, and $p' = p/n$, $q' = q/n$. □

With the usual conventions for curves (for example that (4, 2) means the double of the curve (2, 1)), we can rephrase this as

Corollary 2. *The linear space of quantum observables for $r = 5$ has a basis formed by the identity operator together with the operators* $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, -2)$, $(1, -1)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, -1)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(3, 0)$, $(3, 1)$.

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